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Symmetrical impulsive thermo-fluid dynamic field along a thick plate

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Abstract

The thermo-fluid dynamic field arising on a thick and thermal conductive semi-infinite flat plate is studied when, on one of its sides, a viscous fluid is impulsively accelerated. The case of adiabatic conditions on the unwetted side of the plate is presented. This condition is also representative of symmetrical flow (with respect to the plate axis) around a thick plate with both sides wetted by the fluid. The adopted model, which has been developed in the case of Prandtl number equal to one, is based on an integral formulation of the governing equations. It has been already applied to the case of isothermal condition on the plate side; however main differences characterize the present more complex problem since the governing equations lead to a second-order hyperbolic equation in the space-time variables instead of a first-order one. The solution has been obtained by the Laplace's transformation technique. The effects of the main physical parameters on the temperature behavior at the solid–fluid interface are shown and discussed. The solution accuracy has been verified by comparing the results to those of the limiting case of plate of infinite length.

1. Introduction

In order to analyze the thermo-fluid dynamic boundary layer around a body wetted by a viscous fast fluid, it is necessary to assign a thermal boundary condition along the body. Simple conditions are those of constant temperature or zero heat flux at the wall. A more realistic condition is to impose the continuity of these two functions across the solid–fluid interface; this problem is named conjugated heat transfer. The thermo-fluid dynamic field, studied by means of this condition and coupling the phenomenon of the conduction in the solid body to the conduction and convection in the fluid, was analyzed by many authors for steady regimes. The first studies were those of Luikov [1]; afterwards useful simplifications were suggested, e.g., [2] and, more recently, numerical methods were also used [3].

The main interest in the present problem is driven by the aerospace industry, especially in the study of high-speed aircrafts and rockets, but applications can also be found in other fields such as in the food-freezing industry [4].

Only few works have been written for unsteady conjugated heat transfer problems. An exact solution is given in [5] for the impulsive laminar flow along an infinite thick flat plate. The case of a semi-infinite flat plate, with Prandtl number (Pr) equal to one, was studied in [6] by a low order method and in [7] by a more accurate method taking also into account the energy equation in the fluid. In these works the authors assumed that the unwetted side of the plate is kept at a constant temperature.

In the present paper we study the same impulsive flow with $Pr = 1$ and the same geometry considered in the previous two papers with a different boundary condition: we assume that the unwetted side of the plate is adiabatic. This case is even more useful for the applications because it also includes the case of flow along symmetrical bodies: in fact the symmetry leads to a zero derivative, with respect to the normal axis, of the temperature on the symmetry plane.

We continue to neglect, as generally done in the literature, some important aspects of the phenomenon. For example we do not consider the initial "transitory

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Nomenclature

A, B, C_1, C_2, H, K, L, a, b, c, d_1	constant values
a_i, A_i, $\bar{A}_i$, B_i	coefficients of power series
b	plate thickness
$c_i = \int_0^\infty (1 - Z^i)\,\mathrm{d}z$	
c_n	coefficients of a series
$e_i = 1/[(i+1)!i!]$	
E_1	special function related to the Bessel's function
E_{ij}	constant values
$F(s,\theta) = \mathscr{L}_{\hat{h}}[f(\hat{h},\theta)]$	Laplace's transform of f
$f(\hat{h},\theta) = \exp(\delta\theta + \nu\hat{h})\hat{q}(\hat{h},\theta)$	
$\bar{f} = f_{as}/\hat{h}$	
$G(t) = \int_{-b}^{0} T_s\,\mathrm{d}y$	
$g(s)$, $y(x)$ $Y(x)$	solutions of given ordinary differential equations
g_i	coefficients of the Taylor expansion of u^+
$g_1^*(s)$, $g_2^*(s)$	continuous functions used in the expression of the solution in the asymptotic region
H_{tot}	total enthalpy
H_∞	free-stream total enthalpy
$h(X,\tau)$	scale factor of the dynamic boundary layer
$\hat{h}$, θ	canonical independent variables in the transition region
$I(x)$	invariant of a second-order, linear and homogeneous differential equation
J_q	heat flux at solid–fluid interface
J_1	Bessel's function
$k = \delta\nu$	
L	plate length
$\mathscr{L}_{\hat{h}}$	Laplace's transform
$\mathscr{L}_s^{-1}$	inverse Laplace's transform
$M(a,b,\xi)$	confluent hypergeometric function
M_∞	free-stream Mach number
$p = \frac{b}{L}\frac{\lambda_\infty}{\lambda_s}\sqrt{Re_\infty}$	
Pr	Prandtl number
q_i	coefficients of the Taylor expansion of S^+
$\hat{q} = \hat{h}q_0$	
Re_∞	Reynolds number
$r(s)$	continuous function used in the expression of the solution in the asymptotic region
S	total enthalpy referenced to the free stream value
S^+	non-dimensional total enthalpy
s	independent variable of a given continuous function
s_0	particular value of s
T	temperature
T_{aw}	adiabatic wall temperature in the case of steady flow and $b = 0$
t	time
t_{fs}	ratio between characteristic times of the fluid and solid
U_∞	free-stream velocity
U_n	Heaviside step function
u, v	velocity components
u^+	non-dimensional velocity component u
V	Stewartson–Doronitsyn transformation of normal velocity component v
X	non-dimensional spatial coordinate x
x, y	spatial coordinates
W	Wronskian of a differential equation
W_1, W_2	Wronskian functions of given equations
$Z = \mathrm{er}f(z)$	error function
$z = \frac{\eta\sqrt{Re_\infty}}{Lh(X,\tau)}$	non-dimensional scaled spatial coordinate y

Greek symbols

$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t$	gamma function
α	thermal diffusivity
α_i, β_i, ω_i	coefficients of series
γ	ratio of the specific heats of the fluid
$\delta = 3t_{fs}p^2$	
η	Stewartson–Doronitsyn transformation of spatial coordinate y
κ	constant value
λ	thermal conductivity
μ	dynamic viscosity of the fluid
$\nu = 3/[4(c_2 - c_1)]$	
ξ	independent variable in a given differential equation
ρ	density
τ	non-dimensional time

Subscripts

f	fluid property
s	solid property
X, Z, h, $\hat{h}$, t, x, y, z, η, θ, τ	specify partial derivation respect to the corresponding variable
w	solid–fluid interface condition
∞	free-stream condition

regime" of the present "unsteady" flow: in this way a jump in the temperature at the initial time appears which substitutes the real short transitory regime characterized by an own characteristic time. Its study is beyond the purposes of the present work which instead concentrates on the analysis of the main properties of the physical problem and on the determination of the parameters ruling the phenomenon.

Similar considerations can be made for the leading edge of the plate where a singular behavior exists. Moreover we assume that the thicknesses of the thermal and enthalpic boundary layers are equal in the present case of $Pr = 1$. A more general analysis would be convenient when the influence of the Prandtl number is studied.

Two non-dimensional parameters $t_{\rm fs}$ (ratio of the characteristic times in the fluid and in the solid) and p (related to the thermal conductivities of the solid and of the fluid, to the Reynolds number and to the slenderness of the plate) rule the phenomenon such as for the isothermal condition.

For solving this problem we use the same physical-mathematical model of [7] based on an integral formulation of the boundary layer equations and of the energy equation in the solid in which the axial conduction has been neglected. Unlikely, the simple modification of the boundary condition leads to a final equation governing the problem completely different and more complex than that obtained in the case of constant temperature: in fact now we must solve a second-order hyperbolic partial differential equation instead of a first-order one and its solution required the application of a different mathematical method.

The governing equations are expanded in a power series and the first-order of this expansion has been solved in an exact analytical form. Although here only the first-order solution is studied, the higher-order approximations can be derived and can play an important role in particular conditions as described in the discussion of the method.

The determination of the solution was non-trivial due to the mathematical difficulties induced by the presence of two regions in which the scale factor of the dynamic boundary layer has different expressions. In a so-called "transition" region the solution depends both on space and time, in the "asymptotic" region it only depends on time and the problem is equivalent to the plate of infinite length studied in [5]. The equations have been separately solved in the two zones with a proper boundary condition required for coupling the solutions along the first characteristic line of the momentum equation which specifies the border between the asymptotic and transition regions. The second-order ordinary equation governing the asymptotic region has been solved by the invariant method while the more complex problem of the transition region (second-order partial differential equation) has been solved by Laplace transforms.

The solution accuracy has been verified by comparing present results to the "exact" solution [5] in the asymptotic region.

A number of differences with respect to the previously analyzed case has been found as will be evidenced in the remaining of the paper.

2. Governing equations

The geometry of the flow is sketched in Fig. 1. At the initial time $t = 0$ a fluid at rest with Prandtl number equal to one is impulsively accelerated to a constant speed U_∞ over a semi-infinite, two-dimensional flat plate whose thickness is b. The initial temperature field is uniform in both the fluid and the solid and is $T(x, y, 0^-) = T_\infty$, where the subscript ∞ denotes free-stream conditions. The unwetted plate side is adiabatic. Moreover, in the present model problem, we assume a uniform flow which invests the plate; therefore, at the plate leading edge, a boundary layer with 0 thickness arises. The outer inviscid solution is uniform flow (no shocks or expansions) and, in the boundary layer, the pressure field is also uniform.

We consider here a compressible laminar boundary layer arising near the plate. The flow equations can be simplified by adopting the Stewartson–Doronitsyn transformation

$$\eta = \int_0^y \frac{\rho}{\rho_\infty}\,\mathrm{d}y, \quad V = \frac{\rho}{\rho_\infty} v + \eta_t + u\eta_x \tag{1}$$

with u and v the velocity components, while the subscripts specify partial derivation with respect to the indicated variable. This transformation allows the continuity and momentum equation to be decoupled from the energy equation if $\rho/\rho_\infty = T_\infty/T = \mu_\infty/\mu = \lambda_\infty/\lambda$, where ρ, T, μ, λ are, respectively, the density, temperature, viscosity and conductivity of the

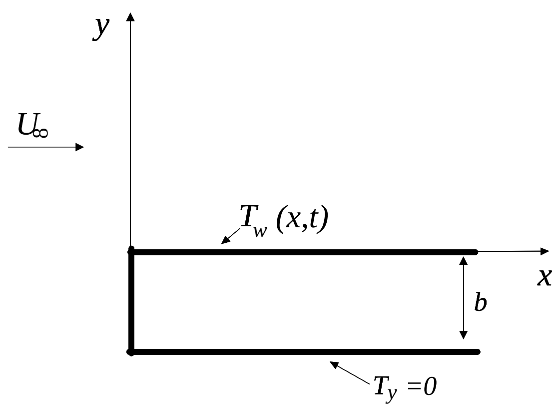

Fig. 1. The geometry of the problem.

fluid. With these assumptions the flow equations reduce to the usual incompressible expression of Prandtl's boundary layer equations that have to be solved together with the heat equation in the solid. On the solid–fluid interface the continuity of the temperature and of the heat flux need to be imposed. The boundary layer equations to be solved are

$$\begin{aligned} &u_x + V_\eta = 0, \\ &u_t + u u_x + V u_\eta = \frac{\mu_\infty}{\rho_\infty} u_{\eta\eta}, \\ &S_t + u S_x + V S_\eta = \alpha_\infty S_{\eta\eta}, \end{aligned} \tag{2}$$

where $S = H_{tot} - H_\infty$ is the total enthalpy referenced to the free stream value and α_∞ is the free-stream thermal diffusivity. The energy equation in the solid is

$$\frac{\partial}{\partial t} T_s = \alpha_s \nabla^2 T_s, \tag{3}$$

where T_s is the temperature in the solid and α_s its diffusivity. The boundary conditions we associate with Eqs. (2) and (3) are given by conditions concerning the fluid field:

$$\begin{aligned} &u(x,\infty,t^+) = U_\infty, \quad S(x,\infty,t) = 0, \\ &u(x,0,t) = V(x,0,t) = 0, \\ &u(x,y,0) = 0, \quad T(x,y,0^-) = T_\infty \end{aligned} \tag{4}$$

on the solid field:

$$T_s(x,y,0^-) = T_\infty, \quad T_{s,y}(x,-b,t^+) = 0 \tag{5}$$

and at the solid–fluid interface:

$$\begin{aligned} &T_s(x,0,t) = T(x,0,t), \\ &\lambda_s \frac{\partial}{\partial y} T_s(x,0,t) = \lambda_f \frac{\partial}{\partial y} T(x,0,t), \end{aligned} \tag{6}$$

where the subscript f specifies fluid property.

The solution method starts [8] from an expansion that reduces from 3 to 2 the number of the independent variables using the Taylor formula in terms of a new variable $Z(\eta)$ for expressing the unknowns. With a suitable approximated expression of the remainder, the Taylor formula gives:

$$\begin{aligned} &u^+(X,Z,\tau) = Z^n + \sum_{i=1}^{n-1} g_i(X,\tau)(Z^i - Z^n), \\ &S^+(X,Z,\tau) = \sum_{i=0}^{n-1} q_i(X,\tau)(Z^i - Z^n), \end{aligned} \tag{7}$$

where $u^+ = u/U_\infty$, $S^+ = S/H_\infty$, $Z(z) = \mathrm{erf}(z)$, (erf denotes the error function), $z = (\eta\sqrt{Re_\infty})/(Lh(X,\tau))$, ($Re_\infty = \rho_\infty U_\infty L/\mu_\infty$ is the Reynolds number), $X = x/L$, $\tau = tL/U_\infty$, L is the axial reference length and

$$g_i(X,\tau) = \frac{1}{i!}\frac{\partial^i u^+}{\partial Z^i}(X,0,\tau), \quad q_i(X,\tau) = \frac{1}{i!}\frac{\partial^i S^+}{\partial Z^i}(X,0,\tau), \tag{8}$$

$h(X,\tau)$ is an unknown scale factor; q_0 and q_1 are related to the unknown temperature and heat flux distributions on the plate wall since $q_0 = S^+_w = T_w/T_{aw} - 1$ and $q_1 = S^+_{Z,w} = (T/T_{aw})_{Z,0}$ with $T_{aw} = T_\infty[1 + (\gamma - 1)/2M^2_\infty]$, where the subscript w specifies conditions at the solid–fluid interface, M_∞ is the free-stream Mach number and T_{aw} is the adiabatic wall temperature in the case of a steady flow over a plate of infinitely small thickness. Finally the integral formulation of the momentum and energy balance Eq. (2) provides the following equations:

$$\frac{\partial}{\partial \tau}\left[h\int_0^\infty (1-u^+)\,dz\right] + \frac{\partial}{\partial X}\left[h\int_0^\infty u^+(1-u^+)\,dz\right] = \frac{1}{h}u^+_{z,w}, \tag{9.a}$$

$$\frac{\partial}{\partial \tau}\left[h\int_0^\infty S^+\,dz\right] + \frac{\partial}{\partial X}\left[h\int_0^\infty u^+ S^+\,dz\right] = -\frac{1}{h}S^+_{z,w}. \tag{9.b}$$

The interested reader is addressed to [7] for a more detailed derivation of the equations that determine the unknown h, g_i and q_i.

In Eq. (7) it is assumed that the thicknesses of the dynamic and enthalpic boundary layers are the same or, equivalently, that the scale factor $h(X,\tau)$ is the same for both layers (this assumption is correct in the case of $Pr = 1$ for the isothermal plate when S is proportional to u).

Sufficiently accurate results can be expected even with $n = 1$. Infact the choice of $Z(z) = \mathrm{erf}(z)$ is not casual since, with $n = 1$, it provides the exact solution of a Rayleigh flow representing the asymptotic behaviour, for $X \to \infty$, of the present flow as it will be discussed in the next section.

This first-order representation, with u^+ and S^+ definitely monotonic, is not accurate in the representation of nearly separating u^+ profiles, which is not the case of the present flow or, in the thermal field, for a plate of infinitely small thickness. Infact, in this case, the adiabatic condition must be imposed at the solid–fluid interface where the S^+ profile is also characterized by an inflection at $Z = 0$. For this condition higher-order approximations are expected to be necessary. In the present case of plate of finite thickness the solid–fluid interface is not adiabatic and we again expect a good modeling by the present method.

To complete the model description we need to consider the energy equation in the solid and formulate the thermal coupling conditions in order to obtain a problem only in terms of unknowns of the fluid field. An integral modeling of the conductive phenomena in the solid plate has been proposed in [5] for $b/L \ll 1$. In this case, by neglecting terms with order larger than $(b/L)^2$ (i.e., the axial conduction in the energy equation for the solid is not taken into account), the heat equation in the solid can be written as

$$\frac{\partial}{\partial t} T_s = \alpha_s \frac{\partial^2 T_s}{\partial y^2}. \quad (10)$$

This equation can accurately describe the thermal field in the solid with the exception of a region very near the leading edge of the plate where the axial conduction cannot be neglected. The Eq. (10) is solved by means of an integral method requiring its integration with respect to y and a suitable assumption on the temperature distribution along y in the solid (see [5] for further details). The integration of this equation with respect to y between $-b$ and 0, together with the second equation in Eq. (5), leads to

$$\frac{\partial}{\partial t} \int_{-b}^{0} T_s \, dy = \alpha_s \left(\frac{\partial^2 T_s}{\partial y^2} \right)_w. \quad (11)$$

By assuming a linear distribution of the thermal gradient in the solid:

$$\frac{\partial T_s}{\partial y} = \left(\frac{\partial T_s}{\partial y} \right)_w \left(1 + \frac{y}{b} \right), \quad (12)$$

we obtain

$$b \left(\frac{\partial T_s}{\partial t} \right)_w - \frac{b^2}{3} \left(\frac{\partial^2 T_s}{\partial y \partial t} \right)_w = \alpha_s \left(\frac{\partial T_s}{\partial y} \right)_w. \quad (13)$$

Eq. (13) can be expressed only in terms of the temperature and its derivatives in the fluid field by imposing at the solid–fluid interface the continuity of the temperature and of the heat flux (Eq. (6)):

$$\frac{3}{b} \frac{\lambda_s}{\lambda_f} T_{\tau,w} - T_{y\tau,w} = 3 t_{fs} T_{y,w}, \quad (14)$$

where

$$t_{fs} = \frac{L}{U_\infty} \frac{\alpha_s}{b^2}$$

is the ratio between the characteristic times of the fluid and of the solid.

3. The first-order solution

The impulsive dynamic boundary layer over the flat plate has been studied by many authors (see [9,10] among others); here we adopted a method which is consistent with the present formulation [11].

A first-order approximation of the velocity and temperature field in the boundary layer is obtained by using $n = 1$ in Eq. (7) only for calculating the left-hand side of Eqs. (9.a) and (9.b), while the right-hand side is expressed directly in terms of the unknowns g_i, q_i.

The integration of the integral momentum Eq. (9.a) provides the scale factor of the dynamic boundary layer

$$h^2(X, \tau) = 4\tau - 8 \left(\frac{\tau}{2} - X \right) U_n \left(\frac{\tau}{2} - X \right), \quad (15)$$

with U_n the Heaviside step function. In order to calculate the thermo-fluid dynamic field this solution is already sufficiently accurate; in [11] the convergence toward the exact solution by increasing n was shown.

In the first-order approximation the dynamic boundary layer is characterized by two regions (see Fig. 2) separated by the characteristic line of the momentum equation $X = \tau/2$. For $X > \tau/2$ ($h = 2\sqrt{\tau}$) the velocity only depends on time (asymptotic region); it is not influenced by the leading edge and describes a Rayleigh-type flow. For $X < \tau/2$ ($h = \sqrt{8X}$) the velocity field only depends on X (steady region) and represents a Blasius-type flow.

In the first-order approximation the energy Eq. (9.b) and the thermal boundary condition Eq. (14) can be written in terms of the unknowns q_0 and q_1 in the following form:

$$\begin{aligned} &(hq_0)_\tau + \frac{(c_2 - c_1)}{c_1} (hq_0)_X = -\frac{2}{h} q_1, \\ &\frac{3}{2c_1 p} q_{0\tau} - \left(\frac{q_1}{h} \right)_\tau = 3 t_{fs} \frac{q_1}{h}, \end{aligned} \quad (16)$$

with $c_i = \int_0^\infty (1 - Z^i) \, dz$ and

$$p = \frac{b}{L} \frac{\lambda_\infty}{\lambda_s} \sqrt{Re_\infty} (c_1 = 1/\sqrt{\pi}, \ c_2 = \sqrt{(2/\pi)}).$$

p and t_{fs} are the non-dimensional parameters ruling the physical problem. In the isothermal case [7] the problem reduced to a first-order hyperbolic equation, on the contrary, we have here a system of two equations that leads to a second-order hyperbolic equation. A new mathematical method of integration is necessary and differences in the physics of the problem can be expected.

In order to associate the boundary conditions with Eq. (16) for the unknowns q_0 and q_1 at $\tau = 0$ and at $X = 0$, a

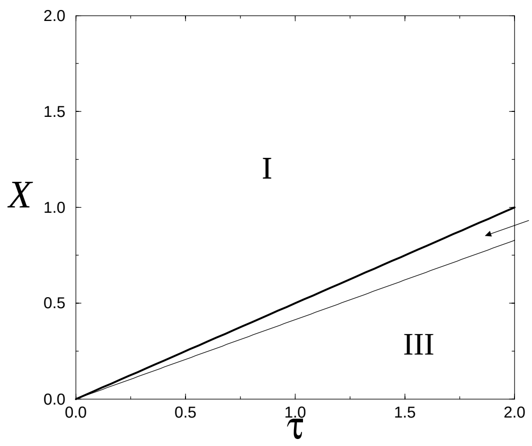

Fig. 2. (X, τ) plane: (I) asymptotic region; (II $\bigcup$ III) transition region. Thick line – characteristic curve of the momentum equation ($X = \tau/2$); thin line – characteristic curve of the energy equation in the transition region ($X = ((c_2 - c_1)/c_1)\tau$).

more detailed analysis of the thermo-fluid dynamic field is necessary and it will be performed in the next two sections. In fact, due to the impulsive character of the flow and to the considered geometry (flat plate) we need to know if the unknowns of the problem, in particular the temperature and the heat flux at the solid–fluid interface are continuous or not. Moreover the thermal field is also characterized in the (X, τ) plane by two regions with different physical properties. It will be shown that the slope of the characteristic lines $(\mathrm{d}X/\mathrm{d}\tau)$ for the energy equation where $h = h(X)$ is lower than that of the momentum equation (see Fig. 2). Therefore, for $X > \tau/2$ both velocity and temperature only depend on time (asymptotic region): in this region they have the same behavior of the case of plate of infinite length. For $X < \tau/2$ the velocity only depends on space while the temperature depends on space and time (transition region).

The integration of the energy equation requires the solution of two different problems for the two regions. In the asymptotic region we shall solve a Cauchy's problem with initial conditions assigned at $\tau = 0$ which is well posed in the whole asymptotic region. In the transition region we shall solve a generalized Goursat problem with one condition given on the characteristic curve (for the energy) $X = 0$ and one condition imposed on the non-characteristic curve (for the energy equation) $X = \tau/2$.

The analysis of the results will be simplified by introducing the similarity variables τ/p^2 and X/p^2 that enables the solution to only depend on one parameter given by the product of t_{fs} and p^2; we put $\delta = 3t_{\mathrm{fs}}p^2$.

4. Asymptotic region

For $X > \tau/2$ the solution is independent of X thus implying that the problem is equivalent to the one studied in [5] (plate of infinite length) where the complete Prandtl's equations were analytically solved. In the semi-infinite plate case it is necessary to find the solution of the same problem at the first-order of our expansion Eq. (7).

By adopting $h = 2\sqrt{\tau}$ as independent variable we can re-write Eq. (16) as follows:

$$(hq_0)_h = -q_1, \quad \frac{3}{2c_1 p} q_{0h} - \frac{1}{h} q_{1h} + \frac{1}{h^2} q_1 = 3t_{\mathrm{fs}} q_1. \tag{17}$$

We derive the boundary conditions in the asymptotic region at $\tau = 0$ by considering the energy balance in the solid, that in an integral form is represented by Eq. (11). It can be written in the form

$$\frac{\partial G(t)}{\partial t} = \alpha_{\mathrm{s}} \left(\frac{\partial T_{\mathrm{s}}}{\partial y} \right)_{\mathrm{w}}, \tag{18}$$

where $G(t) = \int_{-b}^{0} T_{\mathrm{s}}\, \mathrm{d}y$. Our model leads to assume that G is continuous at the initial time: this assumption is compatible with a finite heat flux at the interface, as Eq. (18). Furthermore, using Eq. (12), we obtain

$$\frac{1}{b} G(t) = T_{\mathrm{sw}} - \frac{b}{3} \left(\frac{\partial T_{\mathrm{s}}}{\partial y} \right)_{\mathrm{w}}, \tag{19}$$

which, expressed in terms of q_0 and q_1, becomes

$$\frac{1}{b} G(t) = T_{\mathrm{aw}} \left(1 + q_0 - \frac{2c_1}{3} \frac{q}{h} q_1 \right). \tag{20}$$

For $\tau = 0^-$ the thermal field in the fluid is uniform with the temperature equal to T_∞. It follows:

$$q_0(0^-) = -\frac{(\gamma - 1)M_\infty^2}{[2 + (\gamma - 1)M_\infty^2]} = q_{0\infty}, \quad q_{1h}(0^-) = 0. \tag{21}$$

Hence, for $\tau = 0^+$, the continuity of expression Eq. (20) implies

$$q_0(0^+) - \frac{2}{3} c_1 \frac{q_1(0^+)}{h(0^+)} p = q_{0\infty}. \tag{22}$$

For obtaining a second equation that enables us to determine $q_0(0^+)$ and $q_1(0^+)$ we note that

$$\lim_{h \to 0^+} h q_{0h} = 0. \tag{23}$$

In fact, if this limit would equal a constant κ different than zero, q_0 would locally behave $\kappa \log h$, leading to an infinite temperature along the whole plate at the initial time. It follows, from the first of Eq. (17), that $q_1(0^+) = -q_0(0^+)$. Thus the governing Eq. (17) can be satisfied at $\tau = 0^+$ only if

$$q_0(0^+) = 0, \quad q_1(0^+) = 0. \tag{24}$$

These relations, together with the condition Eq. (22) (for $h \to 0^+$ $q_1(h)/h \to q_{1h}(0^+)$), give

$$q_0(0^+) = 0, \quad q_{1h}(0^+) = -\frac{3}{2c_1 p} q_{0\infty}. \tag{25}$$

It is interesting to note that these conditions lead to a discontinuous temperature on the wall at the initial time with an impulsive variation from T_∞ to T_{aw}. On the contrary q_1 is continuous, although the heat flux (proportional to $q_1/h \to q_{1h}$ for $\tau = 0$) suddenly jumps from 0 to a value given by the initial condition on $q_{1h}(0^+)$.

The boundary conditions Eq. (25) define a well-posed Cauchy problem in the asymptotic region: the solution can be determined starting from $\tau = 0^+$ until the characteristic line of the momentum equation $(X = \tau/2)$ is crossed.

We obtained the solution of Eq. (17) with initial conditions Eq. (25) in analytical form in terms of the confluent hypergeometric function. In Appendix A its derivation is reported; here we summarize the result:

$$q_1 = -\left[C_1 g_1^*(s) + C_2 s r(s) \right],$$

$$\sqrt{\frac{\delta}{2}} \frac{h}{p} q_0 = C_1 \left[s g_2^*(s) - s_0 g_2^*(s_0) \right] + C_2 [1 - r(s)],$$

$$s = s_0 + \sqrt{\frac{\delta}{2}\frac{h}{p}}, s_0 = \frac{3}{2c_1}\frac{1}{\sqrt{\frac{\delta}{2}}} \tag{26}$$

with

$$\begin{aligned} r(s) &= \exp\left[\frac{1}{2}\left(s_0^2 - s^2\right)\right], \\ g_1^*(s) &= \exp\left(-\frac{s^2}{2}\right)M\left(-\frac{1}{2},\frac{1}{2},\frac{s^2}{2}\right), \\ g_2^*(s) &= \exp\left(-\frac{s^2}{2}\right)M\left(\frac{1}{2},\frac{3}{2},\frac{s^2}{2}\right), \\ M(a,b,\xi) &= 1 + \frac{1}{1!}\frac{a}{b}\xi + \frac{1}{2!}\frac{a}{b}\frac{a+1}{b+1}\xi^2 + \dots \end{aligned} \tag{27}$$

and the constants C_1, C_2 (obtained by the initial conditions):

$$\begin{aligned} C_1 &= -\frac{q_{0\infty}}{g_2^*(s_0)} + \frac{C_2}{s_0 g_2^*(s_0)}, \\ C_2 &= \frac{g_1^*(s_0)}{g_2^*(s_0)}\frac{q_{0\infty}}{[s_0 + g_1^*(s_0)/s_0 g_2^*(s_0)]}. \end{aligned} \tag{28}$$

We shall denote by q_{0as} the expression of q_0 thus obtained. An accurate analysis of this solution is not only important for describing the thermo-fluid dynamic field in this region, but also for determining the solution in the transition region: in fact in this region we shall represent q_{0as} in a power series and we need to know the properties of the series as summarized in Appendix A.

The solution is convergent $\forall h$ because the exponential and the confluent hypergeometric function M have both an infinite radius of convergence. For large time values the analytical expression can be simplified by adopting the asymptotic behavior of M (see Eq. (A.13) of Appendix A)

$$s \gg 1 : g_1^*(s) \approx -\frac{1}{s^2}, \quad g_2^*(s) \approx \frac{1}{s^2}. \tag{29}$$

The temperature at the solid–fluid interface (T_w/T_{aw}) is presented in Fig. 3 versus τ/p^2 for different values of δ and $M_\infty = 3$. At the initial time the temperature impulsively reaches T_{aw}, then it decreases until a minimum value; for large values of τ/p^2 it asymptotically approaches the steady regime which is characterized by thermal equilibrium in the solid ($T_s = T_{aw}$) and adiabatic wall conditions at the solid–fluid interface. The minimum wall temperature, the time at which it is reached and the time at which the steady conditions are practically obtained are ruled by the parameter δ.

The non-dimensional heat flux

$$J_q = \frac{p}{\sqrt{Re_\infty}}\frac{LT_{\eta,w}}{T_{aw}} = \frac{2c_1}{(h/p)}q_1 \tag{30}$$

is proposed in Fig. 4, again parametrized with δ.

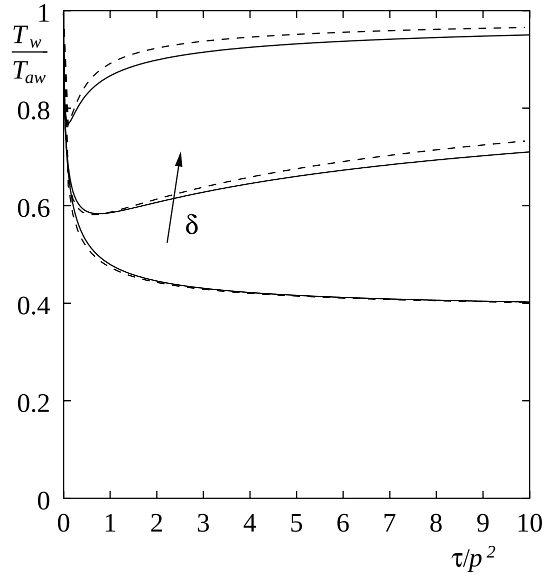

Fig. 3. Asymptotic region. Temperature at the solid–fluid interface versus time. $M_\infty = 3$, $\delta = 0.01$, 1.0, 10. Continuous line – present first-order solution; dashed lines – "exact" reference solution [5].

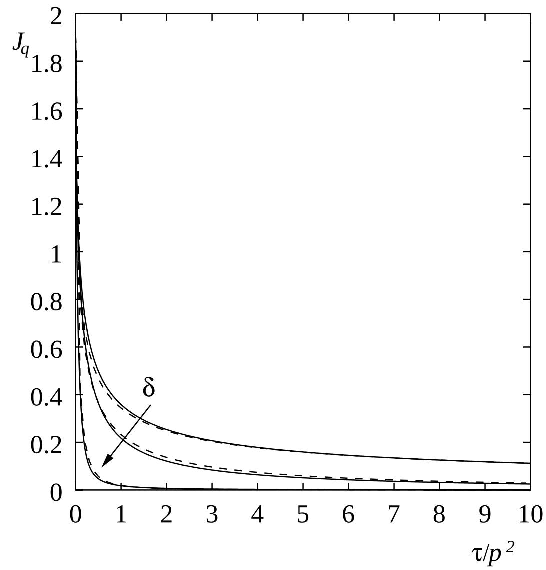

Fig. 4. Asymptotic region. Heat flux at the solid–fluid interface versus time. $M_\infty = 3$, $\delta = 0.01$, 1.0, 10. Continuous line – present first-order solution; dashed lines – "exact" reference solution [5].

In both figures the present results are compared with the "exact" reference solution [5]. The agreement is good with a maximum error of 3% in the wall temperature distribution.

5. Transition region

For $X < \tau/2$ ($h = \sqrt{8X}$) the effects of the leading edge of the plate are relevant and the temperature depends both on X and τ. In this zone there is a transition of the solution from the unsteady behavior to the limiting solution ($\tau \to \infty$) of steady flow over an adiabatic semi-infinite flat plate of infinitely small thickness and thermal equilibrium in the solid plate.

The two families of characteristic curves of Eq. (16) are:

$$X = \frac{c_2 - c_1}{c_1}\tau + \text{const}, \quad X = \text{const}, \tag{31}$$

with $(c_2 - c_1)/c_1 = 1 - \sqrt{2}$. The characteristic line passing through the origin $X = (c_2 - c_1)/c_1\tau$ has a slope ≈ 0.41 which is less than 0.5 (slope of the characteristic for the momentum equation) and is completely contained in the transition region.

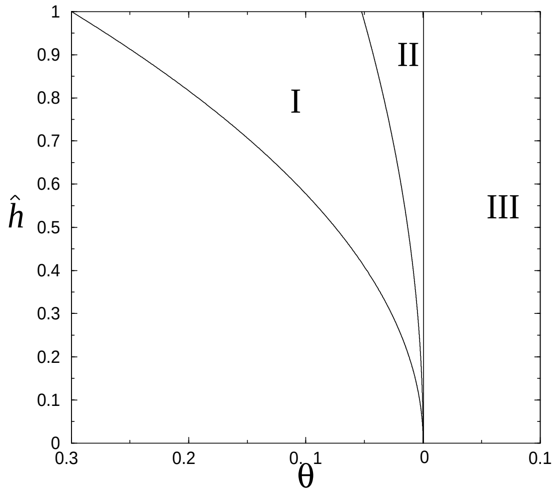

Fig. 5. $(\hat{h}, \theta)$ plane.

5.1. The equations and the boundary conditions

In order to write the Eq. (16) in a simpler form it is useful to adopt the canonical independent variables expressed as follows:

$$\hat{h} = \frac{\sqrt{8X}}{p}, \quad \theta = \frac{\tau}{p^2} - \frac{c_1}{(c_2 - c_1)}\frac{X}{p^2}. \tag{32}$$

Being

$$\frac{\partial}{\partial\tau} = \frac{1}{p^2}\frac{\partial}{\partial\theta}, \quad \frac{\partial}{\partial X} = \frac{1}{p^2}\left(\frac{4}{\hat{h}}\frac{\partial}{\partial\hat{h}} - \frac{c_1}{c_2 - c_1}\frac{\partial}{\partial\theta}\right), \tag{33}$$

the Eq. (16) reduce to:

$$\left(\hat{h}q_0\right)_{\hat{h}} = -\frac{c_1}{2(c_2 - c_1)}q_1, \quad \frac{3}{2c_1}\hat{h}q_{0\theta} - q_{1\theta} = \delta q_1. \tag{34}$$

In Fig. 5 the regions in which the solution develops are mapped into the ($\hat{h}$, θ) plane; in particular the characteristic of the momentum equation separating the asymptotic and transition regions is transformed into the parabola $\theta = -d_1\hat{h}^2$ with $d_1 = [c_1/(c_2 - c_1) - 2]/8$.

By defining as new unknown $\hat{q} = \hat{h}q_0$ and eliminating q_1 in Eq. (34), we obtain the second-order hyperbolic equation governing the problem in the transition region

$$\hat{q}_{\hat{h}\theta} + \delta\hat{q}_{\hat{h}} + v\hat{q}_\theta = 0, \tag{35}$$

with $v = 3/[4(c_2 - c_1)]$.

The solution is uniquely defined by assigning one condition on a characteristic curve ($X = 0$) and one condition on the non-characteristic curve (for the energy equation) $X = \tau/2$ (generalized Goursat problem).

Although at $X = 0$ even the boundary layer Eq. (2) are singular, it is possible to derive, according to these equations, a boundary condition which is sufficiently realistic. In fact, if the temperature at the plate leading edge is finite then ($\hat{h}(X = 0) = 0$)

$$\hat{q}(0, \theta) = \hat{h}(0)q_0(0, \theta) = 0, \tag{36}$$

this is the first boundary condition.

The second condition is obtained by imposing the continuity of the temperature distribution (q_0) on $X = \tau/2$ (coupling of the temperature between the asymptotic and transition region)

$$\hat{q}\left(X = \frac{\tau}{2}, \tau\right) = \hat{h}(\tau)q_{0as}[\hat{h}(\tau)] = 2\sqrt{\frac{\tau}{p^2}}q_{0as}\left(\frac{\tau}{p^2}\right). \tag{37}$$

In the isothermal case [7] the present authors found a singular behavior of the solution in the origin of the (X, τ) plane. It is interesting to verify if a singularity is present in the adiabatic case too. An integration of the first of Eq. (34) leads to

$$q_0 = \frac{\kappa}{\hat{h}} - \frac{c_1}{2(c_2 - c_1)\hat{h}}\int_0^{\hat{h}} q_1 \, d\hat{h}, \tag{38}$$

where κ is an arbitrary constant. A finite value of the temperature at the wall implies $\kappa = 0$ ($q_0 = 0$) and $q_1(0^+, \tau) = 0$ in agreement with relations Eq. (24) valid in the asymptotic region. Thus the singularity is not present in the adiabatic case. The temperature at the leading edge of the plate is constant with respect to the time and equals T_{aw}.

The Eq. (35) can be further simplified by defining

$$f(\hat{h}, \theta) = \exp\left(\delta\theta + v\hat{h}\right)\hat{q}(\hat{h}, \theta) \tag{39}$$

obtaining

$$f_{\hat{h}\theta} - kf = 0, \tag{40}$$

with $k = \nu\delta$ and the boundary conditions given by

$$\hat{h} = 0: f(0,\theta) = \hat{q}(0,\theta)\exp(\delta\theta) = 0,$$
$$\theta = -d_1\hat{h}^2: f(\hat{h}, -d_1\hat{h}^2) = \hat{q}_{as}(\hat{h})\exp\left(-\delta d_1\hat{h}^2 + \nu\hat{h}\right) = f_{as}(\hat{h}), \tag{41}$$

where f_{as} is the function f evaluated in the asymptotic region.

5.2. Solution of the problem

In order to find the function $f(\hat{h},\theta)$ by means of Eq. (40) with boundary conditions Eq. (41) we use the Laplace's transformation. This method, in its standard form, requires the knowledge of the two functions $f(0,\theta)$ and $f(\hat{h},0)$: the first condition is known, but, instead of the second one, we know f along the characteristic curve of the momentum equation. We find the solution in two steps. At first we determine $f(\hat{h},\theta)$ in terms of $f(\hat{h},0)$ by using the Laplace's transformation technique and the boundary condition at $\hat{h}=0$. Subsequently we find the function $f(\hat{h},0)$ by imposing the second boundary condition.

By denoting with $F(s,\theta) = \mathscr{L}_{\hat{h}}[f(\hat{h},\theta)]$ the Laplace's transform of the unknown f, the equation transformed of Eq. (40) reduces to

$$sF_\theta - kF = 0, \tag{42}$$

with solution

$$F(s,\theta) = F(s,0)\left[\exp\left(\frac{k}{s}\theta\right) - 1\right] + F(s,0). \tag{43}$$

The inverse transform of the term within square brackets is [13]

$$\mathscr{L}_s^{-1}\left[\exp\left(\frac{k}{s}\theta\right) - 1\right] = k\theta E_1(-k\theta\hat{h}), \tag{44}$$

where the special function $E_1(\phi)$ is related to the Bessel's function J_1

$$J_1(\phi) = (\phi/2)E_1(\phi/4). \tag{45}$$

By using the convolution theorem we have the following form for the solution of the present problem:

$$f(\hat{h},\theta) = f(\hat{h},0) + k\theta\int_0^{\hat{h}} f(\xi,0)E_1[-k\theta(\hat{h}-\xi)]\,\mathrm{d}\xi, \tag{46}$$

$f(\hat{h},0)$ is computed by imposing the matching with the asymptotic solution.

We represent both $f(\hat{h},0)$ and $f_{as}(\hat{h})$ by means of a power series

$$f(\hat{h},0) = \sum_0^\infty a_i\hat{h}^i, \quad f_{as}(\hat{h}) = \sum_0^\infty A_i\hat{h}^i \tag{47}$$

(note that the radius of convergence of the series representing q_{0as} is infinite).

The matching of the solution on $\theta = -d_1\hat{h}^2$ provides a relation with the following structure

$$\sum_0^\infty A_i\hat{h}^i = \sum_0^\infty a_i\hat{h}^i - kd_1\sum_0^\infty B_i(a_0,\ldots,a_{i-1})\hat{h}^{i+2}. \tag{48}$$

Since the B_i only depend on the a_j with $j < i$ the unknown coefficients a_i can be recursively computed by imposing the equality of the coefficients with the same power of $\hat{h}$:

$$a_i - kd_1B_{i-2} = A_i. \tag{49}$$

The expression of the coefficients A_i and B_i are derived in Appendix B.

5.3. Analysis of the results

The temperature distributions at the solid–fluid interface are displayed in Figs. 6–8, respectively for $\delta = 1$, $\delta = 0.2$ and $\delta = 0.01$ and parametrized for different time values ($M_\infty = 3$). In the case $\delta = 1$ the solution quickly approaches the steady condition of $T_w = T_{aw}$; for smaller values of δ the steady state is reached for much larger τ/p^2 values.

The heat fluxes at the same conditions are presented in Figs. 9–11. In the limits of the present first-order approximation the heat flux is discontinuous at the boundary between the transition and asymptotic regions. It is characterized by a peak value decreasing and moving from the leading edge of the plate as time grows.

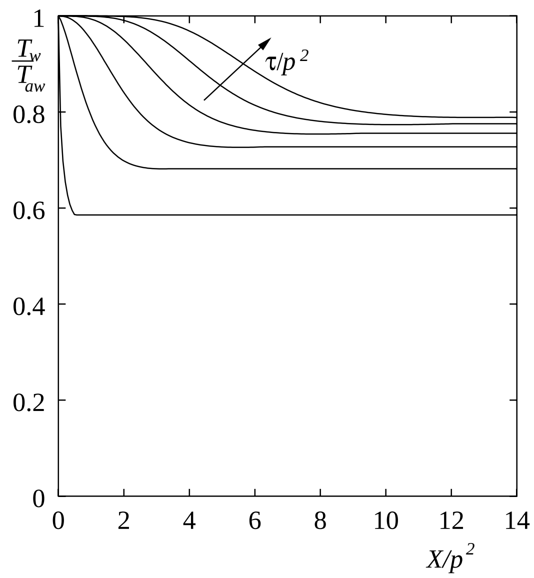

Fig. 6. Transition region, $M_\infty = 3$, $\delta = 1$. Temperature distribution at the solid–fluid interface along the plate. $\tau/p^2 = 1.0$, 6.8, 12.6, 18.4, 24.2, 30.0. Present first-order solution.

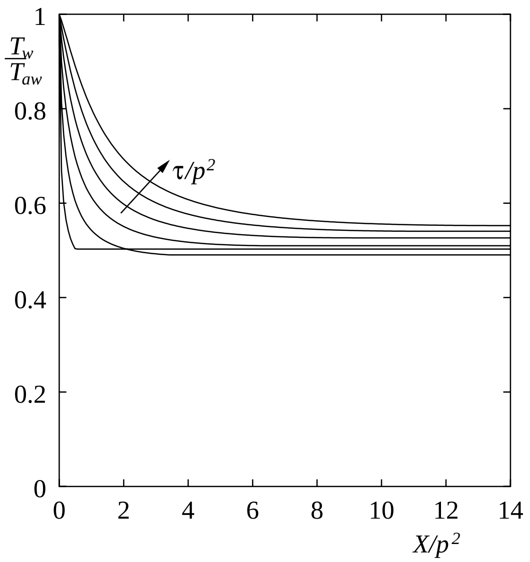

Fig. 7. Transition region, $M_\infty = 3$, $\delta = 0.2$. Temperature distribution at the solid–fluid interface along the plate. $\tau/p^2 = 1.0$, 6.8, 12.6, 18.4, 24.2, 30.0. Present first-order solution.

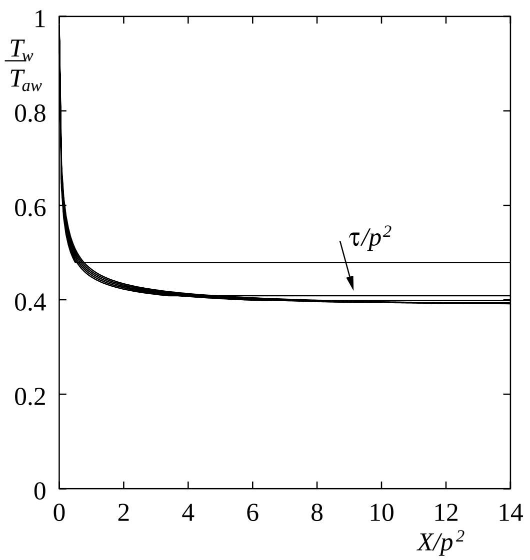

Fig. 8. Transition region, $M_\infty = 3$, $\delta = 0.01$. Temperature distribution at the solid–fluid interface along the plate. $\tau/p^2 = 1.0$, 6.8, 12.6, 18.4, 24.2, 30.0. Present first-order solution.

For small values of δ, which is typical of air–metal interfaces, J_q weakly depends on time in the transition region (see Fig. 11) with a peak value positioned at the leading edge in practice.

As shown in [7], the knowledge of q_0 and q_1 allows the determination of the temperature and velocity profiles in the boundary layer by using any number of terms

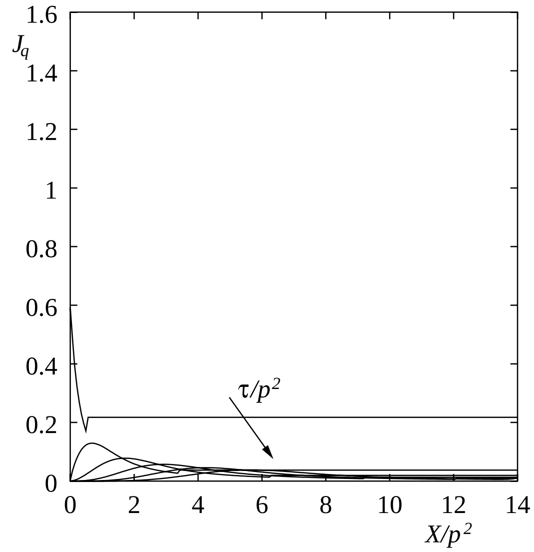

Fig. 9. Transition region, $M_\infty = 3$, $\delta = 1$. Heat flux distribution at the solid–fluid interface along the plate. $\tau/p^2 = 1.0$, 6.8, 12.6, 18.4, 24.2, 30.0. Present first-order solution.

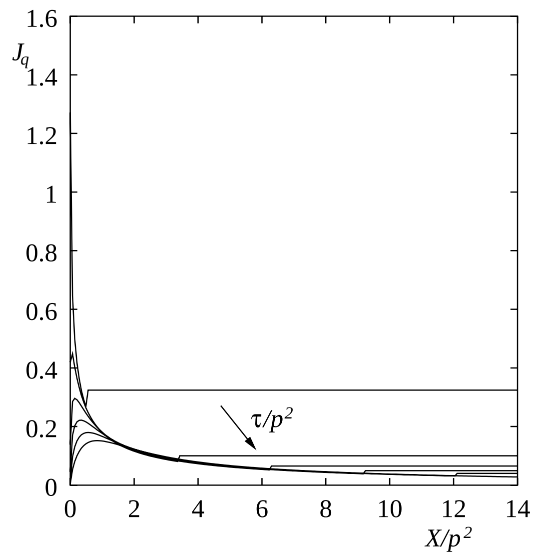

Fig. 10. Transition region, $M_\infty = 3$, $\delta = 0.2$. Heat flux distribution at the solid–fluid interface along the plate. $\tau/p^2 = 1.0$, 6.8, 12.6, 18.4, 24.2, 30.0. Present first-order solution.

in Eq. (7). For instance, a third-order description of the temperature profiles is obtained by the relations

$$
\begin{aligned}
S^+(X,Z,\tau) &= q_0(X,\tau)(1-Z^3) + q_1(X,\tau)(Z-Z^3) \\
&\quad + q_2(X,\tau)(Z^2-Z^3), \\
\frac{T}{T_{\mathrm{aw}}}(X,Z,\tau) &= [1+S^+(X,Z,\tau)] - q_{0\infty}Z^2,
\end{aligned} \tag{50}
$$

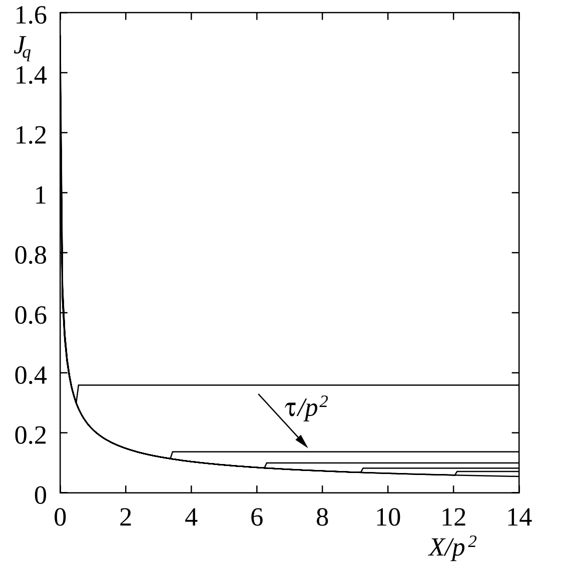

Fig. 11. Transition region, $M_\infty = 3$, $\delta = 0.01$. Heat flux distribution at the solid–fluid interface along the plate. $\tau/p^2 = 1.0$, 6.8, 12.6, 18.4, 24.2, 30.0. Present first-order solution.

with

$$q_2(X,\tau) = \frac{h^2(X,\tau)}{8c_1^2} q_{0\tau}(X,\tau). \quad (51)$$

6. Conclusions

We have presented a first-order solution of the thermo-fluid dynamic problem that arises when a fluid is impulsively accelerated at high speed over a thick semi-infinite flat plate, with its unwetted side adiabatic. This condition of zero heat flux also represents the symmetrical flow around a plate.

The symmetry condition of zero heat flux on the axis of the plate leads to a completely different solution when compared to a previously analyzed case of constant temperature. In fact both the mathematical structure of the governing equation and the physics of the phenomenon changed. The second-order hyperbolic equation, instead of a first-order one, required the development of a new method of solution.

The governing equation has been solved by an exact analytical method based on the Laplace's transformation technique. The task was non-trivial due to the mathematical difficulties induced by the presence of two regions in which the scale factor h of the dynamic boundary layer has different expressions. The equation has been separately solved in the two zones by a proper boundary condition coupling the solutions along the characteristic line of the momentum equation.

The two regions evidenced in the (X, τ) plane are characterized by different properties from a physical viewpoint. For small time values, or equivalently for large X, the flow only depends on time and is not influenced by the plate leading edge (asymptotic region). The problem is here equivalent to the case of a plate of infinite length already solved by an "exact" method. The comparisons of the solutions in this region allowed to verify the good accuracy of the present integral approach. Near the leading edge (with our approximations) or for very large time values, the solutions depend both on space and time (transition region). In this region there are not reference solutions and the capability to analyze this zone is a main contribution of the present work. For infinite values of time the solution in the fluid tends toward the classical case of steady flow over an adiabatic semi-infinite plate.

Two non-dimensional parameters t_{fs} (ratio of the characteristic times in the fluid and in the solid) and p (related to the thermal conductivities of the solid and of the fluid, to the Reynolds number and to the slenderness of the plate) rule the phenomenon such as for the isothermal condition.

Acknowledgements

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Appendix A. Analytical solution in the asymptotic region

With the definition of the new variables

$$s = s_0 + \sqrt{\frac{\delta}{2}}\frac{h}{p}, \quad g(s) = (hq_0)_h, \quad (A.1)$$

where $s_0 = 3/[2c_1\sqrt{\delta/2}]$, the differential problem Eq. (17) with the boundary conditions Eq. (25) reduces to the linear second-order homogeneous equation

$$g'' + sg' + 2g = 0, \quad (A.2)$$

with the initial conditions

$$g(s_0) = 0, \quad g'(s_0) = \frac{3}{2c_1} q_{0\infty}. \quad (A.3)$$

We solved Eq. (A.2) by the invariant method, which is here briefly recalled. The invariant of a general second-order linear and homogeneous differential equation of the type

$$y'' + a_1(x)y' + a_2(x)y = 0 \quad (A.4)$$

is

$$I(x) = a_2(x) - \frac{a_1^2(x)}{4} - \frac{a_1'(x)}{2}. \quad (A.5)$$

By denoting

$$W(x) = \exp\left[-\int a_1(x)\,\mathrm{d}x\right] \tag{A.6}$$

the Wronskian function of Eq. (A.4) and with $Y(x)$ the solution of the equation

$$Y'' + I(x)Y = 0, \tag{A.7}$$

then the solution of Eq. (A.4) is

$$y(x) = W^{1/2}(x)Y(x). \tag{A.8}$$

The invariant and the Wronskian equation of the form (A.2) are

$$I = \frac{3}{2} - \frac{x^2}{4}, \quad W_1 = \exp\left(-\frac{x^2}{2}\right). \tag{A.9}$$

An equation with the same invariant of Eq. (A.9) is the confluent hypergeometric equation

$$\xi y''(\xi) + (b - \xi)y'(\xi) - ay(\xi) = 0, \tag{A.10}$$

which has the solution

$$y(\xi) = AM(a, b, \xi) + B\xi^{(1-b)}M(a - b + 1, 1 - b, \xi), \tag{A.11}$$

where M is the confluent hypergeometric function [12], which can be represented by means of a series as follows:

$$M(a, b, \xi) = 1 + \frac{1}{1!}\frac{a}{b}\xi + \frac{1}{2!}\frac{a}{b}\frac{a+1}{b+1}\xi^2 + \cdots. \tag{A.12}$$

This series has an infinite radius of convergence and its asymptotic behavior is given by

$$\xi \to \infty: \quad M(a, b, \xi) \approx \exp(\xi)\xi^{a-b}\frac{\Gamma(b)}{\Gamma(a)}, \tag{A.13}$$

where Γ is the gamma function.

By defining $s = \kappa\xi^c$ the Eq. (A.10) is transformed into

$$c^2s^2y''(s) + \left[c + b - 1 - \left(\frac{s}{\kappa}\right)^{1/c}\right]csy'(s) - a\left(\frac{s}{\kappa}\right)^{1/c} y(s) = 0, \tag{A.14}$$

which has the solution

$$y(s) = AM\left[a, b, \left(\frac{s}{\kappa}\right)^{1/c}\right] + Bs^{(1-b)/c}M\left[a - b + 1, 1 - b, \left(\frac{s}{\kappa}\right)^{1/c}\right]. \tag{A.15}$$

The invariant of Eq. (A.14) is

$$I(s) = \frac{1}{4s^2}\left(H + Ks^{1/c} + Ls^{2/c}\right), \tag{A.16}$$

where

$$H = -\frac{1}{c^2}(c + b - 1)^2 + \frac{2}{c}(c + b - 1),$$
$$K = \left(\frac{1}{\kappa}\right)^{1/c}\left[-4\frac{a}{c^2} + \frac{2}{c^2}(c + b - 2a) - \frac{2}{c}\right], \tag{A.17}$$
$$L = -\frac{1}{c^2}\left(\frac{1}{\kappa}\right)^{2/c}.$$

By imposing the equality of Eqs. (A.16) and (A.9) we obtain

$$a = -\frac{1}{2}, \quad b = c = \frac{1}{2}, \quad \kappa = \sqrt{2}. \tag{A.18}$$

Since $Y = W_2^{-1/2}y$ (see Eq. (A.8)) with W_2 the Wronskian function of Eq. (A.14), it is possible to obtain the general solution of our problem $g = W_1^{\frac{1}{2}}Y$ where W_1 is specified in relations Eq. (A.9):

$$g(s) = \exp\left(-\frac{s^2}{2}\right)\left[AM\left(-\frac{1}{2}, \frac{1}{2}, \frac{s^2}{2}\right) + Bs\right]. \tag{A.19}$$

The unknowns q_0 and q_1 are related to g by

$$\sqrt{\frac{\delta}{2}}\hat{h}q_0 = \int_{s_0}^{s} g(s)\,\mathrm{d}s, \quad q_1 = -g. \tag{A.20}$$

From the Eq. (A.2) we have

$$\int g(s)\,\mathrm{d}s = -g' - sg + \text{const}, \tag{A.21}$$

hence, taking into account the relation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}M(a, b, \xi) = \frac{a}{b}M(a + 1, b + 1, \xi) \tag{A.22}$$

and computing the constants A and B by the boundary conditions Eq. (A.3) we obtain the solution as specified in the relations Eq. (26).

Appendix B. Derivation of the coefficients A_i and B_i

By eliminating q_1 in the Eq. (17) governing the asymptotic region and defining $\bar{f} = f_{as}/\hat{h}$ we obtain

$$2\hat{h}^2\bar{f}'' + (E_{11}\hat{h} + E_{12}\hat{h}^2 + E_{13}\hat{h}^3)\bar{f}' + (E_{20} + E_{21}\hat{h} + E_{22}\hat{h}^2 + E_{23}\hat{h}^3 + E_{24}\hat{h}^4)\bar{f} = 0, \tag{B.1}$$

where

$$E_{11} = 2, \quad E_{12} = -4v + \frac{3}{c_1}, \quad E_{13} = \delta(8d_1 + 1),$$

$$E_{20} = -2, \quad E_{21} = -2v, \quad E_{22} = 2v^2 + \delta(8d_1 + 1) - \frac{3v}{c_1},$$

$$E_{23} = -\delta(8d_1v - 6\frac{d_1}{c_1} + v), \quad E_{24} = 2\delta d_1(4d_1 + 1).$$

The boundary conditions Eq. (25) in terms of the new unknown $\bar{f}$ are

$$\bar{f}(0) = 0, \quad \bar{f}'(0) = \frac{3}{4c_1} q_{0\infty}. \tag{B.2}$$

Provided the position $\bar{f} = \sum_0^\infty \bar{A}_i \hat{h}^i$, we can recursively compute $\bar{A}_i$ by substituting this relation into Eq. (B.1) obtaining

$$\bar{A}_i = -\Big\{[E_{12}(i-1) + E_{21}]\bar{A}_{i-1} + [E_{22} + E_{13}(i-2)]\bar{A}_{i-2} + E_{23}\bar{A}_{i-3} + E_{24}\bar{A}_{i-4}\Big\}/[2i(i-1) + E_{11}i + E_{20}], \tag{B.3}$$

with $\bar{A}_j = 0 \ \forall j < 0$. The coefficients A_i in $f_{as} = \sum_0^\infty A_i \hat{h}^i$ are obviously given by

$$A_i = \bar{A}_{i-1}. \tag{B.4}$$

The coefficients B_i in Eq. (48) are defined as follows:

$$\int_0^{\hat{h}} f(\xi, 0) E_1[kd_1 \hat{h}^2(\hat{h} - \xi)] \, d\xi = \sum_0^\infty c_i = \sum_0^\infty B_i \hat{h}^i. \tag{B.5}$$

Since the power series expression of the special function E_1 [13] is

$$E_1(\phi) = \sum_0^\infty \frac{(-\phi)^n}{(n+1)!n!}, \tag{B.6}$$

we have the opportunity to solve the integral appearing into Eq. (B.5). In fact

$$\int_0^{\hat{h}} f(\xi, 0) E_1[kd_1 \hat{h}^2(\hat{h} - \xi)] \, d\xi = \int_0^{\hat{h}} \sum_0^\infty a_i \xi^i \sum_0^\infty e_i[-kd_1 \hat{h}^2(\hat{h} - \xi)]^i \, d\xi, \tag{B.7}$$

with

$$e_i = \frac{1}{(i+1)!i!}. \tag{B.8}$$

Furthermore, the Cauchy's theorem on the product of two series

$$\sum_0^\infty \alpha_i \sum_0^\infty \beta_i = \sum_0^\infty \omega_i, \quad \omega_n = \sum_{i=0}^n \alpha_i \beta_{n-i} \tag{B.9}$$

and the relation

$$\int_0^{\hat{h}} \xi^i (\hat{h} - \xi)^{n-i} \, d\xi = \frac{i!(n-i)!}{(n+1)!} \hat{h}^{n+1} \tag{B.10}$$

provide

$$c_n = \sum_{i=0}^n a_i e_{n-i} (-kd_1)^{n-i} \frac{i!(n-i)!}{(n+1)!} \hat{h}^{3n-2i+1}. \tag{B.11}$$

Finally, the comparison of relations Eqs. (B.11) and (B.5) allows the computation of B_i.

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